LOCAL COHOMOLOGY IN FIELD THEORY

with applications to the Einstein equations

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This is an introductory survey of the theory of p-form conservation laws in field theory. It is based upon a series of lectures given at the Second Mexican School on Gravitation and Mathematical Physics held in Tlaxcala, Mexico from December 1–7, 1996. Proceedings available online at http://kaluza.physik.uni-konstanz.de/2MS/ProcMain.html.

1. Introduction.

In these lectures I will provide a survey of some aspects of the theory of local cohomology in field theory and provide some illustrative applications, primarily taken from the Einstein equations of the general theory of relativity. There are many facets to this branch of mathematical physics, and I will only concentrate on a relatively small portion of the subject. A sample of some of the recent literature on the subject can be found in [1].

For the purposes of these lectures I will restrict the development of local cohomology to the theory of p-form conservation laws for field equations. This theory can be considered to be a generalization of Noether's theory of conserved currents to differential forms of any degree. Rather than presenting the theory in its full generality, I will illustrate each point via examples taken from relativistic field theory, with the principal example being the vacuum Einstein equations. In so doing I will summarize the results of a classification of all p-form conservation laws that can be locally constructed from an Einstein metric and its derivatives to any finite order [2]. Finally, I show how the theory and techniques used to analyze p-form conservation laws can be used to give a useful derivation of asymptotic conservation laws (*e.g.*, ADM energy), when they exist, for any field theory.

Many of the basic results on p-form conservation laws can be obtained from a variety of points of view [1]. Virtually all of the results presented here were obtained in collaboration with Ian Anderson. We are currently preparing an in-depth exposition of a theory of p-form conservation laws [2], relative to which these lectures can be considered an introductory survey.

An outline of the topics to be covered is as follows:

- p-form conservation laws in field theory: introduction and examples.
- Jet bundle of metrics and the Euler-Lagrange complex.
- Jet bundle of Einstein metrics.
- p-form conservation laws and local cohomology.
- Identically closed p-forms: cohomology of the free Euler-Lagrange complex.
- Identically closed p-forms locally constructed from a metric: gravitational kink number.
- $(n-1)$-form conservation laws and infinitesimal symmetries of the field equations.

- Lower-degree conservation laws and infinitesimal gauge symmetries of solutions to the field equations.

- Asymptotic conservation laws and local cohomology.

In order to give a reasonably clear picture of the theory without getting mired in technical details, my presentation will be rather informal and lacking in rigor. The reader who wishes to see a more careful and rigorous development of this material can consult the references in [1] and the upcoming publication [2], which I hope should be completed around the time these lecture notes become available.

2. p-form conservation laws in field theory: introduction and examples.

Let M be a differentiable manifold of dimension n and suppose we are given a differential p-form, $p < n$, on M. All geometric objects that we shall use are considered to be (as) smooth (as necessary). To any p-dimensional closed submanifold $\Sigma \hookrightarrow M$, one can associate a real number $Q(\Sigma)$ by integration:

$$Q(\Sigma) = \int_\Sigma \omega.$$

In general, the number $Q(\Sigma)$ is not particularly interesting since it depends on the arbitrary choice of Σ. However, if ω is a *closed p-form*,

$$d\omega = 0,$$

then it follows that $Q(\Sigma)$ is unchanged by continuous deformations of Σ. Conversely, $Q(\Sigma)$ is unchanged by continuous deformations of Σ only if ω is a closed form. More precisely, $Q(\Sigma)$ depends only on the homology class of Σ if and only if ω is a closed form. Of course, if ω is an *exact* form, that is, there exists a $(p-1)$-form α such that

$$\omega = d\alpha,$$

then because $d^2 = 0$ it follows that ω is closed, but by Stokes' theorem it follows that $Q(\Sigma) = 0$ for any choice of $\Sigma \hookrightarrow M$. Thus one can use the equivalence classes of closed modulo exact forms to study the homological properties of M, and this is the basis of

de Rham cohomology. All of these notions can be generalized to open manifolds (that is, those with asymptotic regions) and/or manifolds with boundary provided one imposes appropriate boundary conditions.

In field theory one, in effect, uses de Rham cohomology classes to represent *p-form conservation laws*. Roughly speaking, a p-form conservation law is an assignment of a closed spacetime p-form $\omega[\varphi]$ to each solution φ of the field equations. Given a p-form conservation law, one can associate a number $Q[\varphi]$ to each solution of the field equations by integrating the p-form over an appropriate p-dimensional submanifold Σ. Because the resulting number is unchanged by continuous deformations of the embedding of Σ in spacetime, we say that the number is a "conserved quantity". Indeed, when Σ is, or is contained in, a spacelike hypersurface, the invariance of Q with respect to continuous deformations of Σ corresponds to the familiar notion of a "constant of the motion". One way to make a p-form conservation law for any field theory is to associate an exact p-form to all solutions of the field equations. Of course, in this case $Q[\varphi] = 0$ for all solutions. Consequently, we say that such p-form conservation laws are *trivial*. We are therefore interested in cohomology classes, that is, equivalence classes of maps from the space of field configurations into closed modulo exact p-forms on spacetime. In this sense the study of p-form conservation laws is a branch of local cohomology in field theory. Despite their similarities, local cohomology in field theory is rather different than de Rham cohomology of manifolds. The de Rham cohomology involves equivalence classes of closed forms modulo exact forms on a manifold, while the local cohomology studies equivalence classes of maps from field configurations into closed spacetime p-forms. In effect, local cohomology in field theory involves a theory of *local formulas* for defining closed forms from spacetime field configurations. The de Rham cohomology detects global properties of manifolds; in the small, the de Rham cohomology is always trivial. Local cohomology can be non-trivial even when one works in the small.

At this point it is worth making clear the way we are using the word "local". By far the most common use of the term is the one which signifies "in the small" or "in a sufficiently small neighborhood". This is *not* the use of the word in "local cohomology", or in "... locally constructed from the field...". This use of the word "local" signifies that the

objects under consideration depend, at a given point q in spacetime, upon the fields and their derivatives to some finite order at q. Thus, the Einstein Lagrangian density $\sqrt{|g|}R$ is a "local density" in the sense that it is a scalar density which is constructed locally from the metric and its first two derivatives. Local here does not mean, of course, in the small since the Lagrangian density is globally well-defined. By contrast, the value of a vector at a point p obtained by parallel propagation of a given vector at a point q is *not* a local metric quantity in the sense in which we want to use the word "local". This is, of course, because the value of the vector at p depends upon the metric and its first derivatives along the curve connecting q and p. Thus the connotation of local in "local cohomology" is not that the considerations are only valid in the small, but rather that the objects being used depend in a local fashion on the fields of interest.

The above notion of p-form conservation laws will become considerably clearer if we introduce some examples.

3-form conservation law: charged scalar field in 4 dimensions.

Consider a complex scalar field φ on a 4-dimensional spacetime (M, g_{ab}) satisfying the Klein-Gordon equation

$$\nabla_a \nabla^a \varphi - m^2 \varphi = 0. \tag{2.1}$$

We define the 3-form

$$\omega = \epsilon_{abcd} J^d dx^a \wedge dx^b \wedge dx^c \tag{2.2}$$

where

$$J^d = i(\varphi^* \nabla^d \varphi - \varphi \nabla^d \varphi^*).$$

The form ω and the vector field J^d are locally constructed from the Klein-Gordon field in the sense that at a given point they depend upon only φ and $\nabla_a \varphi$ *at that point*. It is a nice exercise to check that the exterior derivative of ω vanishes by virtue of the field equation (2.1) and its complex conjugate. This is equivalent to the statement that the current J^a is divergence-free "on-shell", which is very straightforward to verify from the identity:

$$\nabla_d J^d = i(\varphi^* \nabla^d \nabla_d \varphi - \varphi \nabla^d \nabla_d \varphi^*)$$
$$= i[\varphi^* (\nabla_a \nabla^a \varphi - m^2 \varphi) - \varphi(\nabla_a \nabla^a \varphi^* - m^2 \varphi^*)].$$

With appropriate boundary conditions, the integral of ω over a spacelike hypersurface defines a conserved $U(1)$ charge for the field theory which can be coupled, *e.g.*, to an electromagnetic field.

In field theory conservation laws usually arise via conserved 3-forms in 4 spacetime dimensions (or conserved $(n-1)$-forms in n dimensions), such as in the example above. But as the following examples show, there are also a variety of important "lower-degree" conservation laws.

2-form conservation law: electromagnetic field.

The electromagnetic field can be described by an antisymmetric tensor field, the "field strength tensor" F_{ab}, built from a one-form A_a via

$$F_{ab} = \nabla_a A_b - \nabla_b A_a.$$

In regions of spacetime not containing electromagnetic sources the field strength satisfies the field equations

$$\nabla^b F_{ab} = 0. \tag{2.3}$$

In 4 spacetime dimensions, a 2-form conservation law for these field equations is provided by

$$\omega = \epsilon_{abcd} F^{cd} dx^a \wedge dx^b. \tag{2.4}$$

It is a straightforward exercise to verify that ω in (2.4) is closed when the field equations (2.3) are satisfied. The integral of ω over a spacelike two-sphere yields the electric charge contained in the two-sphere. This is, of course, just a version of Gauss' law, which we see can be interpreted as a 2-form conservation law.

2-form and 1-form conservation laws: vacuum spacetimes with a Killing vector.

Consider vacuum regions of spacetime satisfying the Einstein equations

$$G_{ab} = 0, \tag{2.5}$$

and which admit a Killing vector field, that is, a vector field K^a such that

$$\nabla_{(a} K_{b)} = 0. \tag{2.6}$$

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Associated with these equations there is a 2-form conservation law, the Komar 2-form,

$$\kappa = \epsilon_{abcd}\nabla^c K^d dx^a \wedge dx^b, \tag{2.7}$$

and a 1-form conservation law, the "twist" 1-form,

$$\tau = \epsilon_{abcd}K^b\nabla^c K^d dx^a. \tag{2.8}$$

To verify that these are indeed conservation laws, simply take their exterior derivative and you will find, with a little algebra, that the result is algebraic in the Einstein tensor G_{ab}, the Lie derivative $\mathcal{L}_K g_{ab}$, and its derivative $\nabla_c \mathcal{L}_K g_{ab}$. The integral of the Komar 2-form over a closed spacelike surface, *e.g.*, a sphere in the vacuum region surrounding a star, defines conserved energy, momentum, or angular momentum, depending upon the nature of the Killing vector field.

0-form conservation law: dilaton gravity in two dimensions.

Here the spacetime is two-dimensional and the fields are a metric g_{ab} and a scalar field φ. Using the derivative ∇ compatible with the metric, the field equations are

$$R[g] + \lambda'(\varphi) = 0, \tag{2.9}$$

and

$$\nabla_a\nabla_b\varphi - \frac{1}{2}\lambda(\varphi)g_{ab} = 0, \tag{2.10}$$

where $R[g]$ is the scalar curvature of g_{ab} and $\lambda(\varphi)$ is a local function of φ. These field equations admit a 0-form conservation law

$$\alpha = \nabla^a\varphi\nabla_a\varphi - \int d\varphi\lambda(\varphi). \tag{2.11}$$

The 0-form α is a local function of the fields and their first derivatives which becomes a constant when the field equations are satisfied. This is easily verified by computing

$$\nabla_a\alpha = 2(\nabla_a\nabla_b\varphi - \frac{1}{2}\lambda(\varphi)g_{ab})\nabla^b\varphi.$$

The on-shell (constant) value of α corresponds, *e.g.*, to the "mass" of the dilatonic black hole [3]. The existence of a 0-form conservation law is somewhat remarkable in a field theory

since such conservation laws normally arise as integrals of motion in mechanical systems with a finite number of degrees of freedom (which can be thought of as field theories in one spacetime dimension). The conservation of α in dilaton theories reflects the fact that the pure dilaton gravity theory (with no matter couplings) has only a finite number of degrees of freedom once the action of the gauge group (spacetime diffeomorphisms acting by pullback on the the fields) has been factored out. Indeed, up to a diffeomorphism, the fields φ and g_{ab} solving (2.9), (2.10) can be obtained by solving a system of ordinary differential equations for which the 0-form conservation law plays the role of Hamiltonian. In other words, the dilaton field equations are, modulo gauge, equivalent to a finite-dimensional dynamical system. This result can be understood from a simple counting of constraints in the Hamiltonian formulation of the theory. There are 2 first-class constraints associated with the field equations (2.9) and (2.10). Because there are four fields in the theory, (g_{ab}, φ), there are no field degrees of freedom left over after factoring out the gauge transformations.

Asymptotic conservation laws: the ADM energy.

Our final example is the ADM energy, which is an instance of an *asymptotic conservation law*, that is, a *p*-form whose integral in an asymptotic region is conserved by virtue of field equations and (asymptotic) boundary conditions. The ADM energy in general relativity is defined via a surface integral at infinity in an asymptotically flat spacelike hypersurface [4]. One expression of it is

$$E_{ADM} = \frac{1}{16\pi} \lim_{r \to \infty} \sum_{i,j=1}^{3} \int_{t,r=\text{const.}} (g_{ij,j} - g_{ii,j}) \, d^2 S^j. \tag{2.12}$$

Here the indices i and j refer to an asymptotically Cartesian coordinate chart (t, x^i) on an asymptotically flat spacelike hypersurface $t = \text{constant}$. The coordinate r is the associated (asymptotic) radial variable and the integral is over a coordinate 2-sphere with coordinate area element $d^2 S^j$. Given asymptotically flat boundary conditions on the spacetime metric, it can be shown that E is independent of the choice of t and of the choice of asymptotically Cartesian coordinates whenever the metric solves the vacuum Einstein equations.

This final example is not, strictly speaking, a *p*-form conservation law of the same type as those presented above, *e.g.*, because it is not displayed as a closed spacetime differential

form. However, we shall show later that it can nevertheless be understood using elements of our theory of p-form conservation laws.

Hopefully, these examples demonstrate that p-form conservation laws are ubiquitous and important in field theory. The basic purpose of this course is to give an introduction to a subset of techniques that have been developed recently to (i) explain the existence of these conservation laws field theoretically, and (ii) systematically find these conservation laws, when they exist, for any field theory. Of course, for the conservation laws associated with conserved currents (e.g., 3-form conservation laws in 4-dimensional spacetime), point (i) is addressed via the Noether theory relating symmetries of a Lagrangian to conservation laws, but even for conserved currents results on point (ii) are probably less familiar to you (though they are well-developed, see [19]). For lower-degree conservation laws, points (i) and (ii) are only now being developed by mathematicians and physicists [1,2]. In essence, my goal here is to describe machinery that allows one to find a generalization of Noether theory to all p-form conservation laws.

I would like to make this course a relatively accessible introduction to the theory of p-form conservation laws. Since the predominant background of the participants of this school is in general relativity, I will eschew a general, abstract treatment and develop the ideas largely in the context of a familiar and important example: the vacuum field equations of Einstein's general theory of relativity.

Our first step is to give a more precise characterization of what we mean when we say "p-form conservation law", which formalizes the basic properties of the examples we have just discussed. This is most easily done using the language of jets.

3. Jet bundle of metrics and the Euler-Lagrange complex.

The mathematical setting for the theory of p-form conservation laws in a field theory for a metric is the *jet bundle of metrics* $\mathcal{J}$. Given a spacetime manifold M and local coordinates x^i, a point $p \in \mathcal{J}$ is specified by giving a spacetime point x_0^i, a metric at that point g_{ij} (a symmetric, non-degenerate rank-two tensor with the appropriate signature), and a sequence of tensors $g_{ij,k}$, $g_{ij,kl}$, ... representing the value of the derivatives of the

metric at x_0^i. We write

$$p = (x_0, g_{ij}, g_{ij,k_1}, \ldots, g_{ij,k_1 \ldots k_n}, \ldots) \in \mathcal{J}. \tag{3.1}$$

In this way a local coordinate chart on M defines a chart on $\mathcal{J}$. It can be shown that an atlas on M then leads to a well-defined global construct, namely $\mathcal{J}$, which is suitably independent of the choice of atlas. Then, as you might expect, it is possible to give an intrinsic, coordinate-free definition of $\mathcal{J}$. See [5,19] for details. For simplicity, we will be content with a purely local, coordinate-based treatment, which will be adequate for the level of presentation of these lectures. The space $\mathcal{J}$ so-constructed is a fiber bundle $\mathcal{J} \rightarrow M$. $\mathcal{J}$ can be viewed as a bundle $\mathcal{J} \rightarrow E$, where $E \rightarrow M$ is the *bundle of metrics*. Given a chart on M, a point in E is specified by giving the pair (x_0^i, g_{ij}). A metric tensor field $g_{ij}(x)$ defines a cross-section of both E and $\mathcal{J}$. Conversely, given a point p in $\mathcal{J}$ (or E), one can always find a smooth metric tensor field which takes the values defined by p.

It is often convenient to label points in $\mathcal{J}$ in a way that is better adapted to the geometric meaning associated to the metric and its derivatives [6]. To this end, let Γ^i_{jk} be the Christoffel symbols of the metric g_{ij}, and let us recursively define "prolonged Christoffel symbols", which are to be viewed as functions on the jet bundle, by

$$\Gamma^i_{j_0 j_1 \cdots j_k} = \Gamma^i_{(j_0 j_1 \cdots j_{k-1}, j_k)} - (k-1)\Gamma^i_{m(j_1 \cdots j_{k-2}} \Gamma^m_{j_{k-1} j_k)}, \quad k = 1, 2, \ldots \quad . \tag{3.2}$$

I will schematically denote these variables by $\Gamma^{(k)}$ ($\Gamma^{(1)}$ denotes the usual Christoffel symbol). The $\Gamma^{(i)}$ $i = 1, \ldots, l$ are algebraically independent at any given spacetime point, and capture all of the spacetime coordinate information that is hiding in the first l derivatives of the metric. Roughly speaking, given metric components $g_{ij}(x)$ in some chart, the $\Gamma^{(k)}$ measure the difference between the given coordinates x^i on spacetime and a geodesic coordinate system for the metric $g_{ij}(x)$. In particular, all of the $\Gamma^{(k)}$ variables vanish at the origin of a geodesic coordinate system. Next, let us examine the spacetime geometric content of the jet bundle. Of course we expect geometrical information to enter through the curvature tensor and its covariant derivatives. But these quantities do not give a good parametrization of the jet bundle because they are subject to the Ricci and Bianchi identities as well as identities obtained from these by differentiation, and so these variables are

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not algebraically independent at a given point in spacetime. To handle these identities we introduce the following tensors, which are being viewed as functions on the jet bundle,

$$Q_{ij,j_1 \cdots j_k} = g_{ir} g_{js} \nabla_{(j_3} \cdots \nabla_{j_k} R^r{}_{j_1}{}^s{}_{j_2)}, \quad k = 2, 3, \ldots \quad . \tag{3.3}$$

Here R_{abcd} is the Riemann tensor. These tensors have the symmetries

$$Q_{ij,j_1 \cdots j_k} = Q_{(ij),j_1 \cdots j_k} = Q_{ij,(j_1 \cdots j_k)},$$

and

$$Q_{i(j,j_1 \cdots j_k)} = 0,$$

and can be shown to be algebraically independent at any given point of spacetime. We will schematically denote these variables by $Q^{(k)}$. Note that the variables $Q^{(i)}$, $i = 2, \ldots, l$ depend upon the first l derivatives of metric. The variables $Q^{(k)}$ contain all spacetime-geometric information defined by a point in the jet bundle. These variables were used by Penrose [7] and are closely related to Thomas's "normal metric tensors" [8]. It can be shown that the variables

$$(x^i, g_{ij}, \Gamma^{(1)}, \Gamma^{(2)}, \ldots, Q^{(2)}, Q^{(3)}, \ldots) \tag{3.4}$$

uniquely parameterize points in the jet bundle [6]. Put differently, the variables (3.4) are the freely specifiable data at a point of a (pseudo-)Riemannian manifold.

The reason for introducing the notion of a jet bundle is that it is a natural arena for building local functions of the metric and its derivatives. So, for example, the scalar curvature of a metric,

$$R[g] = R(g_{ij}, g_{ij,k}, g_{ij,kl}),$$

can be viewed in an obvious way as a map from $\mathcal{J}$ into the real numbers. More importantly for our purposes, we can define a *p-form locally constructed from the metric*, $\omega[g]$ as a map from $\mathcal{J}$ into the (bundle of) p-forms on M depending upon an arbitrary but finite number of derivatives of g_{ij}. We write

$$\omega[g] = \omega_{i_1 \ldots i_p}(x, g, \partial g, \cdots, \partial^k g) dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

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As you can easily see, all of our previous examples of p-form conservation laws, in particular, the Komar and twist conservation laws, can be viewed as such maps from an appropriate jet bundle into the spacetime p-forms. More generally, a tensor field $T[g]$ locally constructed from the metric is a map from $\mathcal{J}$ into the relevant bundle of tensor fields. Given a specific metric tensor field, that is, a cross section of the bundle of metrics E, we have a corresponding cross section of $\mathcal{J}$ which can be used to pull back $T[g]$ to a tensor field on M. This fancy sounding construction simply means that we substitute a given metric into the formula for $T[g]$.

Next we need to introduce an appropriate notion of derivatives of functions on jet space. To begin, let $f[g]$ be a 0-form locally constructed from the metric, that is, $f[g]$ associates a number to each point in $\mathcal{J}$. Again, the scalar curvature of the metric is an example of such an object. We define the *total derivative*, $D_i f$, of $f[g]$ using the chain rule:

$$D_i f = \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial g_{kl}} g_{kl,i} + \ldots + \frac{\partial f}{\partial g_{kl,i_1 \ldots i_p}} g_{kl,i_1 \ldots i_p i} + \ldots \quad .$$

This is the field-theoretic analog of the "total time derivative" in particle mechanics. Given a metric tensor field, $g_{ij}(x)$, we have a cross section $\sigma \colon M \to \mathcal{J}$; the total derivative is defined so that

$$(D_i f) \circ \sigma = \frac{\partial}{\partial x_i} (f \circ \sigma),$$

or, in an alternative notation,

$$\left((D_i f)[g] \right)_{g = g(x)} = \frac{\partial}{\partial x_i} (f[g(x)]).$$

Note that the total derivative of a function locally constructed from the first k derivatives of the metric is a function locally constructed from the first $k+1$ derivatives of the metric. We can now define the (total) exterior derivative of a p-form $\omega[g]$ locally constructed from the metric:

$$D\omega[g] = D_{[k} \omega_{i_1 \ldots i_p]}(x, g, \partial g, \ldots, \partial^l g) dx^k \wedge dx^{i_1} \wedge \ldots dx^{i_p}.$$

Evidently, $D\omega$ is a $(p+1)$-form locally constructed from the metric. It is a simple exercise to check that $D^2 = 0$. Thus the spaces $\Omega^p(\mathcal{J})$ of p-forms locally constructed from the metric form a *complex*, analogous to the de Rham complex. If the spacetime dimension is

n, then there will be no non-zero p-forms with $p > n$. In particular, if $\omega[g]$ is an n-form, then $D\omega \equiv 0$. It would appear that the complex thus stops at $\Omega^n(\mathcal{J})$. But it turns out that there is a useful (infinite) continuation of the spaces $\Omega^p(\mathcal{J})$ which defines the *Euler-Lagrange complex* [9]. This complex gets its name from the fact that the next operator in the sequence defines the Euler-Lagrange equations from a Lagrangian (viewed as a 4-form).

Although we shall not really need to use the full Euler-Lagrange complex, I can briefly expand a little on the basic ideas. To begin, an n-form $\lambda[g]$ locally constructed from the metric can be viewed as defining a Lagrangian; the corresponding action functional S is obtained via an integral over an n-dimensional region $\mathcal{R} \subset M$ of spacetime:

$$S = \int_{\mathcal{R}} \lambda[g].$$

Let us extend our space of differential forms to include differentials of the field variables $(g_{ij}, g_{ij,k}, \ldots)$ (*i.e.*, we consider differential forms on $\mathcal{J}$). The Euler-Lagrange equations $E^{ij}(\lambda)$ are obtained by applying a differential operator on $\mathcal{J}$—the Euler-Lagrange operator $E^{ij}(\cdot)$—to the Lagrangian. The resulting collection of functions of the metric and its derivatives defines an $(n + 1)$ form on $\mathcal{J}$:

$$E(\lambda) = E^{ij}(\lambda)dg_{ij}.$$

We thus can define a new space $\Omega^{n+1}(\mathcal{J})$. The image of the Euler-Lagrange operator on n-forms defines elements of $\Omega^{n+1}(\mathcal{J})$, but this operator is neither surjective or injective. In particular, if $\alpha[g]$ is an $(n-1)$-form, then $D\alpha$ is an n-form, *i.e.*, a Lagrangian, whose Euler-Lagrange expression is trivial: $E^{ij}(D\alpha) \equiv 0$. This is just the differential form expression of the familiar result that the Euler-Lagrange equations are trivial if the Lagrangian density is a total divergence. Thus a non-trivial extension of the operators

$$D\colon \Omega^p(\mathcal{J}) \to \Omega^{p+1}(\mathcal{J}),$$

satisfying $D^2 = 0$, to the case where $p = n$ is to replace D with the Euler-Lagrange operator acting upon Ω^n. It is this element of the sequence that gives the Euler-Lagrange complex its name. Since we will only be discussing the p-form conservation laws, the extension of

the complex for p-forms of degree $p \geq n$ will not interest us. However, this extension is quite important in field theory. For example, the operator following the Euler-Lagrange operator in the sequence annihilates all $(n + 1)$-forms that are in the image of the Euler-Lagrange operator. The cohomology at this form degree can then be profitably used to study the inverse problem in the calculus of variations [9].

We now can give a more precise definition of a "closed p-form locally constructed from the metric". It is simply an element $\omega[g] \in \Omega^p(\mathcal{J})$ satisfying $D\omega = 0$. However, we have not yet injected into the discussion the idea that the p-form should be closed when the metric satisfies the field equations. To do this we need to find out what it means to go "on-shell" using the jet bundle.

4. Jet bundle of Einstein metrics.

Given a chart on spacetime, we can view the Einstein tensor as 10 functions on $\mathcal{J}$. In particular, the vacuum Einstein equations

$$G_{ab} = 0$$

define 10 relations between the metric and its first two derivatives, *i.e.*, a subspace of $\mathcal{J}$. Of course, any solution to the field equations will also solve all equations obtained by taking all derivatives of the field equations, and these equations provide additional relations in $\mathcal{J}$. Thus solutions to field equations define points on the *Einstein equation manifold*, $\mathcal{E} \in \mathcal{J}$, which is the set of points in $\mathcal{J}$ satisfying the relations dictated by the Einstein equations and all of their derivatives. Of course, it is not obvious that $\mathcal{E}$ does form a manifold, but it can be shown that $\mathcal{E}$ is in fact a smooth sub-bundle of $\mathcal{J}$ [6]. By the way, do not confuse $\mathcal{E}$ with the space of solutions to the Einstein equations. Solutions to the Einstein equations define cross sections of $\mathcal{E}$, just as metric tensor fields define cross-sections of the (jet) bundle of metrics E $(\mathcal{J})$. In particular, unlike $\mathcal{E}$, the space of solutions to the Einstein equations need not be a manifold [10].

Conservation laws only need hold modulo the field equations, or as one often says, "on shell". What this means is that the total exterior derivative of a p-form conservation law, which is a $(p + 1)$-form-valued function on $\mathcal{J}$, must vanish when evaluated on points

in $\mathcal{E}$. By the same token, any two p-form conservation laws that are equal on shell should be considered physically equivalent. In effect, what this means is that p-form conservation laws are determined by their restriction to $\mathcal{E}$. For these reasons it is often quite useful to have an explicit parametrization of $\mathcal{E}$. Such a parametrization can be found in [6,13] and has proven useful in a variety of applications, so we briefly summarize it here.

Because the Einstein equations only restrict the geometry of spacetime, they do not place any restrictions upon the jet coordinates $(x^i, g_{ij}, \Gamma^{(k)})$, $k = 1, 2 \ldots$. The second derivatives of the metric are restricted via the vanishing of the Ricci tensor, or, equivalently, all traces of $Q^{(2)}$ must vanish. Remarkably, this pattern continues to all the higher derivatives. More precisely, the vacuum Einstein equations and all of their derivatives completely fix the traces of the variables $Q^{(k)}$ in terms of their completely trace-free parts, which we shall denote by $\widetilde{Q}^{(k)}$:

$$\widetilde{Q}_{ij,j_1 \cdots j_k} = g_{ir} g_{js} \nabla_{(j_3} \cdots \nabla_{j_k} R^r{}_{j_1}{}^s{}_{j_2)} - (\text{all traces}), \quad k = 2, 3, \ldots . \tag{4.1}$$

Thus the variables

$$(x^i, g_{ij}, \Gamma^{(1)}, \Gamma^{(2)}, \ldots, \widetilde{Q}^{(2)}, \widetilde{Q}^{(3)}, \ldots) \in \mathcal{E} \tag{4.2}$$

parametrize points in $\mathcal{E}$, *i.e.*, are the freely specifiable data at a point of an Einstein spacetime.

The coordinates (4.2) on the equation manifold $\mathcal{E}$ can be interpreted in terms of a power series expansion of an Einstein metric. More precisely, if we are trying to build an Einstein metric by Taylor series we (i) specify the spacetime point x^i around which the series is being developed, (ii) specify the metric components g_{ij} at x^i, (iii) specify the variables $\Gamma^{(k)}$; this fixes the coordinate system in which the metric components are being built, (iv) specify the variables $\widetilde{Q}^{(k)}$, which supplies the geometric content of the Einstein metric. from this freely specifiable data, the Einstein equations can be used to construct all derivatives of the metric at the chosen point, which defines the formal power series solution to the field equations. Of course, this procedure leaves open the question of convergence of the series.

The parametrization (4.2) turns out to be somewhat unwieldy in applications, primarily because of the need to remove so many traces. A much more useful parametrization

uses a spinor representation of the variables $\widetilde{Q}^{(k)}$ [6,7,11,13]. The spinor representation of $\mathcal{E}$ that arises in 4 spacetime dimensions is as follows. Let Ψ_{ABCD} and $\overline{\Psi}_{A'B'C'D'}$ denote the Weyl spinors (see [12] for definitions). Fix a soldering form $\sigma_a^{AA'}$ such that, for a given g_{ij},

$$g_{ij} = \sigma_i^{AA'}\sigma_{jAA'}.$$

It can be shown that the variables $Q^{(k)}$ are uniquely parametrized by the soldering form, the spinor variables

$$\Psi^{(k)} \longleftrightarrow \Psi_{J_1\cdots J_{k+2}}^{J'_1\cdots J'_{k-2}} = \nabla_{(J_1}^{(J'_1}\cdots\nabla_{J_{k-2}}^{J'_{k-2})}\Psi_{J_{k-1}J_kJ_{k+1}J_{k+2})} \tag{4.3}$$

and their complex conjugates $\overline{\Psi}^{(k)}$. Thus we obtain a spinor parametrization of the Einstein equation manifold in terms of

$$(x^i, g_{ij}, \Gamma^{(1)}, \Gamma^{(2)}, \ldots, \Psi^{(2)}, \overline{\Psi}^{(2)}, \Psi^{(3)}, \overline{\Psi}^{(3)}, \ldots). \tag{4.4}$$

This spinor representation of $\mathcal{E}$ proved essential in the classification of symmetries of the Einstein equations (see [11,13] and §8 below).

5. p-form conservation laws and local cohomology.

We are now ready to give a precise definition of a p-form conservation law. We consider a p-form $\omega[g]$, locally constructed from the metric, to be a map from $\mathcal{J}$ into the spacetime p-forms depending upon a finite but arbitrary number of derivatives of the metric. We say that $\omega[g]$ is a *p-form conservation law* if

$$D\omega = 0 \qquad \text{on } \mathcal{E}.$$

Of course, as we have noted before, if $\omega[g]$ is an exact p-form then it is of no interest. We therefore say that a p-form conservation law is *trivial* if there is a $(p-1)$-form $\alpha[g]$, locally constructed from the metric and its derivatives to some finite order, such that

$$\omega[g] = D\alpha[g].$$

Any two p-form conservation laws that differ by a trivial conservation law on-shell will define the same conserved quantity; consequently it is useful to identify any two conservation laws that differ by an exact form on-shell:

$$\omega \sim \omega'$$

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if there exists $\alpha[g]$ such that

$$\omega = \omega' + D\alpha \quad \text{on } \mathcal{E}.$$

Thus we are lead to consider on-shell equivalence classes $[\omega]$ which define the *local cohomology* of the field theory. This cohomology can be obtained by pulling back the Euler-Lagrange complex from $\mathcal{J}$ to $\mathcal{E}$ using the natural embedding of $\mathcal{E}$ into $\mathcal{J}$. Even if there is no cohomology in the Euler-Lagrange complex associated to the jet bundle $\mathcal{J}$ (*i.e.*, there are no topological conservation laws, see §6), there *can* be cohomology (*i.e.*, p-form conservation laws) in the Euler-Lagrange complex associated to the equation manifold $\mathcal{E}$. This situation is somewhat analogous to the fact that, *e.g.*, all closed forms are exact on $\mathbf{R}^3$, but this is not so on the torus $\mathbf{T}^2 \hookrightarrow \mathbf{R}^3$. A better, rather elementary example of this phenomenon, taken from field theory, is as follows. Let the field theory of interest be a that of a scalar field φ on an n-dimensional spacetime. It can be shown (see [14] and/or §6 below) that identically closed p-forms, with $0 < p < n$ locally constructed from φ are necessarily exact. On the other hand, the following $(n-1)$-form is closed on the equation manifold defined by the wave equation $\nabla_a \nabla^a \varphi = 0$:

$$\omega[\varphi] = \epsilon_{a_1 \ldots a_n} \nabla^{a_n} \varphi \, dx^{a_1} \wedge \ldots \wedge dx^{a_{n-1}}. \tag{5.1}$$

But there is no $(n-2)$-form locally constructed from φ whose exterior derivative yields $\omega[\varphi]$ on shell. Analogous remarks apply to the conserved form (2.2) for the charged scalar field.

It is important to stress at this point that even if the de Rham cohomology of the spacetime manifold is completely trivial, there can still be interesting local, field theoretic cohomology. So, for example, suppose the free scalar field theory, from which (5.1) is derived, is formulated on $\mathbf{R}^n$. If we evaluate the $(n-1)$-form (5.1) (or (2.2) in the charged case) on any particular solution $\varphi(x)$ of the field equations then the resulting closed form $\omega(x) = \omega[\varphi(x)]$ on M is guaranteed to be exact (as a form on M) since all closed forms are exact on $\mathbf{R}^n$. This does not alter the fact that (5.1) defines a non-trivial cohomology class, that is, a p-form conservation law, for the field theory. The triviality on M of the form $\omega(x)$ does *not* imply the triviality of the form $\omega[\varphi]$ on $\mathcal{E}$ since the $(n-2)$ form $\alpha(x)$ on M which satisfies $\omega(x) = d\alpha(x)$ cannot be constructed locally from the field and its

derivatives. For this reason, field theoretic cohomology is sensitive to structures such as conservation laws without the need to specify global data such as boundary conditions on solutions.

6. Identically closed p-forms: cohomology of the free Euler-Lagrange complex.

Before investigating methods for computing closed p-forms locally constructed from solutions to the field equations, it is worth pausing to first understand *identically* closed p-forms, that is, p-form conservation laws that arise irrespective of field equations. As we shall see, such p-form conservation laws reflect topological properties of the bundle of fields. Consequently, we shall refer to equivalence classes of such conservation laws (modulo exact forms) as *topological conservation laws.*

We now cite, without proof and somewhat informally, some basic results concerning the origins of topological conservation laws. Again, for simplicity, we phrase the results in terms of our prime example: general relativity, but it is not hard to see how to generalize these results to other field theories.

Our first result is that there is an isomorphism between the cohomology classes of the Euler-Lagrange complex and that of the de Rham complex on $\mathcal{J}$. This result is proven, *e.g.*, in [9] using standard techniques from homological algebra in the variational bicomplex. Next, it can be shown that the de Rham cohomology of the jet bundle $\mathcal{J}$ is isomorphic to that of the bundle of metrics E. Thus, modulo exact p-forms locally constructed from a metric, all identically closed p-forms which are constructed locally from a metric can be obtained by finding non-trivial cohomology classes in E. The construction of a closed form on M from a closed form on E is simple to exhibit in local coordinates; it amounts to pulling back the form on E by a cross-section. Recall that coordinates on E are (x^i, g_{ij}), where x^i are coordinates on spacetime. Thus, for example, a differential 1-form α on E can be written as

$$\alpha = A_i(x, g)dx^i + B^{ij}(x, g)(x, g)dg_{ij}.$$

The corresponding form $\alpha[g]$ on M is obtained by

$$\alpha[g] = [A_i(x, g) + B^{jk}(x, g)g_{jk,i}]dx^i.$$

It is a simple exercise to show that if α is a closed form on E, then $\alpha[g]$ is a closed form on M locally constructed from the metric. It is also a simple exercise to generalize this example to any p-form; one simply pulls back the form from E to M using an arbitrary local section and reinterprets the result as an element of the Euler-Lagrange complex. The main result, then, is that modulo exact forms all identically closed p-forms locally constructed from a metric arise this way. Thus, topological conservation laws for a metric theory can always be expressed via the metric and its first derivatives only.

If the bundle of fields is a vector bundle, then all cohomology of the bundle comes from lifting closed forms from the base space (the spacetime manifold) [25]. These forms, of course, have nothing to do *per se* with the field theory. Thus in this case there will be no interesting topological conservation laws (*cf.* [14]). On the other hand, the bundle of metrics is not a vector bundle, and there *do* exist topological conservation laws for field theories based upon pseudo-Riemannian metrics.

7. Identically closed p-forms locally constructed from a metric: gravitational kink number.

To compute topological conservation laws in metric theories of gravity, we need to understand the cohomology of the bundle of metrics E over the spacetime manifold M. I remind you that this is not the infinite-dimensional space of metrics, but rather is the finite dimensional bundle (dim $E = 14$ when dim $M = 4$) whose cross sections are metric tensor fields. In coordinates, a point in E is a pair (x^i, g_{ij}), where x^i labels a point in the spacetime manifold and g_{ij} are components of a symmetric, non-degenerate rank-2 tensor with the appropriate signature. We will assume M is orientable and the metric is Lorentzian, *i.e.*, it has signature $(-+++\ldots)$.

It is shown by Steenrod [15] that the bundle of Lorentzian signature metrics over an n-dimensional spacetime manifold admits a deformation retraction to a bundle E' over spacetime whose typical fiber is diffeomorphic to the real projective space RP^{n-1} (which can be defined as the space of lines through the origin in $\mathbf{R}^n$), so that it is the cohomology of *this* bundle that we need to compute. The origin of the real projective space is fairly easy to understand. It is a standard result from Lorentzian geometry (see e.g., [16]) that a

manifold admits a Lorentzian metric if and only if it admits a *line element field* (or *direction field*), which is a continuous assignment of a vector, up to a non-zero multiplicative factor, to each point of the manifold. This line element field is, at each point of M, a line in the tangent space to the point which lies inside the light cone. To compute this line in a coordinate chart at any given point, one simply finds the eigenvector with negative eigenvalue of the matrix of components of the metric in that chart at the point of interest. With Lorentzian signature, the resulting vector is unique up to a multiplicative factor. Thus the metric defines, at each point, an element of RP^{n-1}. In fact, it can be shown that the metric defines a global mapping from spacetime into RP^{n-1}. The map is not canonical however; it depends upon other data besides just the manifold and metric.

It can be shown [17] that the cohomology of E' comes from either the base manifold or from the fibers. The cohomology of M leads to conservation laws that have nothing to do with the metric. Only the fiber cohomology leads to interesting topological conservation laws in the sense that it leads to conserved p-forms locally constructed from metric. So we need the cohomology of RP^{n-1}, which is given by constants at degree zero and volume forms at degree $n-1$ when n is even. When n is odd, RP^{n-1} is not orientable and the only cohomology classes arise at degree zero, so that the only conserved quantities are the rather uninteresting constant functions. Thus, non-trivial topological conservation laws arise when the spacetime dimension is even, and are obtained by pulling back a volume form on RP^{n-1} to spacetime using the line element field construction [17]. The resulting topological conservation law is of degree $n-1$, and so corresponds to a topological current. The conserved quantity obtained by integrating the normal component of this current over a hypersurface is the "kink number" conservation law, first obtained by Finkelstein and Misner in 1959 using homotopy considerations [18]. In even-dimensional, orientable spacetimes the kink number is the degree of the map from a hypersurface to RP^{n-1} which is provided by the metric. Somewhat more figuratively, the kink number measures the number of times a light cone tumbles as one moves along the hypersurface. This kinking of the light cone field defines an integer. A continuous deformation of the hypersurface induces a continuous change in the metric evaluated on the surface, which, however, cannot induce a continuous change in an integer. Thus the kink number is conserved. It can be

shown [18] that if the kink number is non-zero then there will be no hypersurfaces that are everywhere spacelike (in the homology class with non-zero kink number).

To summarize, in even dimensional, orientable spacetimes, the kink number of Finkelstein and Misner can be interpreted as arising from an identically closed $(n-1)$-form locally constructed from the metric. Modulo exact forms, this is the only topological conservation law for metric field theories. Let me add that this kink conservation law does not arise if the metric is Riemannian, since in this case E retracts to M [15]. The reason for this is clear: the kink conservation law arises because of the possibility of distinguishing a timelike direction at each point of spacetime. This possibility does not arise with Riemannian metrics.

8. $(n-1)$-form conservation laws and infinitesimal symmetries of the field equations.

Now we consider the problem of finding and explaining $(n-1)$-form conservation laws (n is the spacetime dimension) associated to field equations such as the Einstein equations. Given a metric on spacetime, such conservation laws are equivalent to the existence of conserved currents via the Hodge duality operation. Thus we are on the familiar territory usually handled via Noether's theorem [19]. Consequently, nothing I will reveal here is new, field-theoretically speaking, except perhaps the presentation, which is not so traditional in the physics literature. My mode of presentation is designed to jibe with an analogous treatment of lower-degree conservation laws in subsequent sections, which *is* new. Again, it is convenient to present the discussion in the context of our prime example, the vacuum Einstein field equations.

So, consider a spacetime $(n-1)$-form locally constructed from the metric which is closed when the vacuum Einstein equations hold. This is equivalent to the existence of a spacetime vector field $J^a[g]$ locally constructed from the metric which is divergence-free on-shell,

$$\nabla_a J^a[g] = 0 \qquad \text{on } \mathcal{E}.$$

Here the covariant divergence is built from the total derivative in the obvious way. Because the Einstein equation manifold $\mathcal{E}$ is a submanifold of $\mathcal{J}$, it can be shown that this

conditional equality is equivalent to the existence of tensor-valued functions, ρ_{ab}, ρ_{abc}, ...,
on $\mathcal{J}$ such that there is an *identity* on $\mathcal{J}$

$$\nabla_a J^a[g] = \rho_{ab} G^{ab} + \rho_{abc} \nabla^c G^{ab} + \dots , \tag{8.1}$$

where the number of terms on the right-hand side of (8.1) depends upon the derivative
order of the metric appearing in the current J^a. Of course, the current is only defined by
this relation up to the possible addition of an identically conserved current. For example,

$$J_0^a[g] = \nabla_b S^{ab}[g], \qquad S^{ab} = -S^{ba}, \tag{8.2}$$

is divergence-free identically, *i.e.*, it satisfies (8.1) with all the multipliers ρ vanishing.
In the language of differential forms, the conserved current J_0^a is dual to an exact $(n-1)$-form, and so is uninteresting. Ignoring topological conservation laws, which we have
already enumerated for this theory, the only ambiguity in J^a is that associated with trivial
conservation laws (8.2).

The identity (8.1) can be considerably simplified since we are only interested in the
on-shell values of the current J^a. For example, note that the second term on the right
hand side of (8.1) can be written as

$$\rho_{abc} \nabla^c G^{ab} = \nabla^c (\rho_{abc} G^{ab}) - (\nabla^c \rho_{abc}) G^{ab}. \tag{8.3}$$

The first term in (8.3) can be absorbed into an on-shell trivial redefinition of J^a. The
second term can be absorbed into a redefinition of the multiplier ρ_{ab}. It is easy to see that
a similar trick works on all higher-derivative terms in (8.1). Thus, up to on-shell trivial
redefinitions of J^a, we can consider the simpler identity

$$\nabla_a J^a[g] = \rho_{ab}[g] G^{ab}. \tag{8.4}$$

In differential form language, this is the same as

$$D\omega[g] = \Lambda_{ab}[g] G^{ab}, \tag{8.5}$$

where $\omega[g]$ is an $(n-1)$-form and $\Lambda_{ab}[g]$ is a symmetric-tensor-valued n-form, both locally
constructed from the metric and its derivatives. Our task is to find all identities of the
form (8.4) or (8.5). Notice that the obvious integrability condition for (8.5),

$$D(\Lambda_{ab} G^{ab}) = 0$$

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is identically satisfied since $\Lambda_{ab}G^{ab}$ is a spacetime form of maximal degree. One might be tempted to think that (8.5) must therefore always have solutions, which is of course false. Keep in mind that the equations (8.4-5) are differential equations on $\mathcal{J}$, not on the spacetime manifold M. The correct integrability condition for (8.5) is that the Euler-Lagrange operator annihilate the "trivial Lagrangian n-form" $\Lambda_{ab}G^{ab}$. We have already indicated this fact when discussing the structure of the Euler-Lagrange complex in §3. Now, one can compute in a straightforward, albeit lengthy, manner the restrictions placed upon Λ_{ab} or ρ_{ab} by this integrability condition. Keep in mind, though, that *a priori* the multipliers depend upon an arbitrary but finite number of derivatives of the metric. A useful short cut, based upon the relation of the Euler-Lagrange operator to the variational calculus, is as follows.

Integrate both sides of (8.4) over a region $B \subset M$:

$$\int_B \sqrt{|g|}\nabla_a J^a[g] = \int_B \sqrt{|g|}\rho_{ab}G^{ab}. \tag{8.6}$$

Because the integrand on the left-hand-side of (8.6) is a total divergence, we get

$$\int_{\partial B} \sqrt{|\gamma|}n_a J^a[g] = \int_B \sqrt{|g|}\rho_{ab}G^{ab}, \tag{8.7}$$

where n_a is the unit normal to the boundary ∂B, which we assume is non-null, and γ_{ab} is the induced metric on ∂B. Consider a field variation δg_{ab} of compact support. For a suitably "large" region B the boundary ∂B will be outside the support of δg_{ab}. The change in the left-hand side of (8.7) induced by this variation vanishes, so that we conclude

$$\int_B \delta\left(\sqrt{|g|}\rho_{ab}G^{ab}\right) = \int_B \left(\sqrt{|g|}G^{ab}\delta\rho_{ab} + \rho_{ab}\delta(\sqrt{|g|}G^{ab})\right) = 0 \tag{8.8}$$

for variations δg_{ab} of compact support. Because the Einstein equations come from varying an action, *i.e.*, are the Euler-Lagrange equations for some Lagrangian, the linearized Einstein operator

$$L^{ab}(\delta g) = \delta(\sqrt{|g|}G^{ab})$$

is formally self-adjoint (see, *e.g.*, [19,20]):

$$\int_B \rho_{ab}L^{ab}(\delta g) = \int_B \delta g_{ab}L^{ab}(\rho).$$

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Using (8.8) we find that

$$\int_B \delta g_{ab} L^{ab}(\rho) = 0$$

whenever $G_{ab} = 0$ and δg_{ab} is of compact support. By the fundamental theorem of the variational calculus, we conclude that

$$L^{ab}(\rho) = 0 \quad \text{on } \mathcal{E}. \tag{8.9}$$

To summarize: every conserved current has an associated solution $\rho_{ab}[g]$ of the linearized equations locally constructed from the fields and their derivatives. Such a solution to the linearized equations defines an infinitesimal *generalized symmetry* transformation of the field equations in the sense that, if $g_{ab}(x)$ is a solution to the Einstein equations, then to first order in ϵ so is $g_{ab}(x) + \epsilon \rho_{ab}[g(x)]$. We have arrived at one version, actually a sort of converse, of Noether's theorem: *Associated to every conserved current of a Lagrangian field theory is a symmetry transformation of the field equations.* Topological conservation laws, such as the kink current, have $\rho_{ab} = 0$, which *is* a symmetry albeit a trivial one. Note, however, that not every symmetry of the field equations necessarily is associated to a conserved current via (8.4). A symmetry of the field equations is necessary but not sufficient for the existence of the conserved current. As an elementary example, consider a massless scalar field φ satisfying the wave equation

$$\nabla^a \nabla_a \varphi = 0.$$

It is easy to see that for any constant c the multiplier

$$\rho[\varphi] = c\varphi$$

satisfies the linearized field equation (which is of course just the wave equation again) when φ satisfies its field equation. This solution of the linearized equations reflects the scaling symmetry of the wave equation. However, there is no current locally constructed from the field and its derivatives such that

$$\nabla_a J^a = c\varphi \nabla_b \nabla^b \varphi,$$

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so that there is no conservation law associated with the scaling symmetry. Of course, a necessary and sufficient condition is that the symmetry preserve a Lagrangian for the field equations. This is the basis of the Noether theory [19]. Typically, scaling symmetries of field equations do not preserve an underlying Lagrangian.

Thus an integrability condition for the existence of a conserved $(n-1)$-form as in (8.5) is the existence of a solution to the linearized field equations locally constructed from the fields. It is not hard to see that this result is not specific to the Einstein equations, but will arise for any variational system of field equations.

Remark:

For linear field equations, schematically,

$$L(\varphi) = 0,$$

every solution $\rho[\varphi]$ of the field equations built locally from a solution φ to the field equations has a corresponding conservation law. This is just an application of the identity

$$\rho[\varphi]L(\varphi) - \varphi L(\rho) = \nabla_a J^a[\varphi],$$

which holds for any linear differential operator L for some vector field J^a locally constructed from φ (see *e.g.*, [20]). This identity does not, however, guarantee that the conservation law is non-trivial. For example, the scaling symmetry of the wave equation corresponds, in the above sense, to an *identically* conserved current, *i.e.*, an exact (hence trivial) $(n-1)$-form.

It is a routine (albeit sometimes challenging) affair to compute generalized symmetries for field equations [19]. For example, the generalized symmetries of the vacuum Einstein equations (in four spacetime dimensions) have been classified as follows [11,13]. Let $\rho_{ab}[g]$ be a tensor-valued function on $\mathcal{J}$ which satisfies the linearized Einstein equations on $\mathcal{E}$. Then, modulo terms that vanish on $\mathcal{E}$ (*i.e.*, on shell trivial solutions), there is a constant c and a vector field $X^a[g]$ locally constructed from the metric so that

$$\rho_{ab} = cg_{ab} + \nabla_{(a}X_{b)}. \tag{8.10}$$

The first term represents the scaling symmetry $g \to cg$ admitted by the vacuum equations, while the second term reflects the diffeomorphism symmetry of the field equations. Neither the scaling symmetry nor the diffeomorphism symmetry are associated with non-trivial conserved currents. In detail, the scaling symmetry does not preserve a Lagrangian, and the diffeomorphism symmetry corresponds to an on-shell trivial conserved current:

$$J^a = X_b G^{ab}.$$

This strongly suggests, but does not quite prove, that there are no non-trivial conservation laws for the Einstein equations (aside from the kink conservation law). The difficulty is that we lack a strict 1–1 correspondence between on-shell symmetries and conserved currents, the latter being defined by off-shell identities (8.4). It *is* nevertheless true that there are no non-trivial, non-topological conservation currents for the vacuum Einstein equations, but the proof is rather involved: it requires computing all conserved p-forms of the linearized field equations, and the use of homological algebraic constructions from the variational bicomplex [9] that are beyond the scope of this survey. A crucial ingredient in this classification of conserved $(n-1)$-forms is a classification of conservation laws of lower form degree. This is our next topic.

9. Lower-degree conservation laws and infinitesimal gauge symmetries of solutions to the field equations.

Our analysis of conserved $(n-1)$-forms, also known as conserved currents, led us to an analysis of symmetries of the field equations, by virtue of integrability conditions of (8.5). Our analysis of lower-degree conservation laws will follow a similar strategy. Remaining in the context of our Einstein equations example, we begin with

$$D\omega[g] = \rho_{ab}[g]G^{ab} + \rho_{abc}[g]\nabla^c G^{ab} + \cdots + \rho_{abc\cdots r}[g]\nabla^c \cdots \nabla^r G^{ab}, \qquad (9.1)$$

where $\omega[g]$ is a representative of a p-form conservation law with $0 \le p < n-1$, and the multipliers $(\rho_{ab}, \rho_{abc}, \ldots)$ are tensor-valued $(p+1)$-forms locally constructed from the metric. The integrability conditions for solutions to this equation are

$$D\left(\rho_{ab}[g]G^{ab} + \rho_{abc}[g]\nabla^c G^{ab} + \ldots + \rho_{abc\cdots r}[g]\nabla^c \cdots \nabla^r G^{ab}\right) = 0. \qquad (9.2)$$

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The analysis of these conditions is rather different, and somewhat more involved, than that which arose in the last section. Various results on lower-degree conservation laws, obtained using a variety of technologies, can be found in [1]. A general theory of lower-degree conservation laws, and in particular equations such as (9.2), will be given a detailed, rigorous presentation in a forthcoming paper [2]. In these lectures I will simply illustrate a typical set of results of the analysis via the Einstein equations in four spacetime dimensions. It is not too hard to infer the basic features of the generalization of the gravitational results to different dimensions and other field theories.

Because the identities (9.2) must hold everywhere in $\mathcal{J}$, the integrability conditions reduce to a hierarchy of algebraic and differential conditions on the multipliers ρ. These are the lower-degree conservation law analogues of (8.9). Using these conditions it can be shown that, up to on-shell trivial redefinitions of the conservation law ω, that is, up to terms that are exact when the field equations hold, the identity can be reduced to the form

$$D\omega[g] = \rho_{ab}[g]G^{ab}. \tag{9.3}$$

Equation (9.3) is the lower-degree conservation law analog of (8.5). It should be kept in mind that the form ω and the multiplier ρ_{ab} have been redefined in passing from (9.1) to (9.3). In order to keep the notation uncluttered, we retain the original symbols. It should be noted that, unlike the analysis of closed $(n-1)$-forms, the "integration by parts" simplification of (9.1) to (9.3) is *not* in general guaranteed for lower-degree conservation laws. The existence of such a simplification depends upon details of the symbol of the field equations. For gauge theories of the type usually considered the simplification is available.

We can now begin again with the analysis of the integrability conditions for (9.3). The integrability conditions for these equations again involve algebraic and differential conditions on the multipliers $\rho_{ab}[g]$. The algebraic conditions force ρ_{ab} to vanish if ω is a 0-form or 1-form. Thus we immediately conclude that there are only topological conservation laws at form degree 0 and 1. But there are no interesting topological conservation laws at degree 0 and 1 (see §7). We thus see that the vacuum Einstein equations admit no conserved 0-forms or 1-forms.

If ω is a 2-form, the same algebraic conditions that ruled out conserved 0 and 1-forms

imply the existence of a vector field $X^a[g]$, locally constructed from the metric and its derivatives, such that

$$\rho_{ab} = X_{(a}\epsilon_{b)cde}dx^c \wedge dx^d \wedge dx^e.$$

The remaining integrability conditions involve differential conditions upon the 3-form multiplier ρ_{ab} which are equivalent to the total differential equation

$$\nabla_a X_b + \nabla_b X_a = 0, \quad \text{on } \mathcal{E}. \tag{9.4}$$

This formally looks like the Killing equation, and in a certain sense it is, but we should be careful to understand it more precisely. Eq. (9.4) says that there is a vector field locally constructed from the metric and its derivatives such that the symmetrized total covariant derivative vanishes when the Einstein tensor and all its derivatives vanish. This is, of course, not quite the same as asking for a Killing vector field, which is a spacetime vector field related to a specific metric tensor field via the Killing equation. Here we need a map from the jet space of Einstein metrics into the space of vector fields on M that satisfies a particular differential equation in $\mathcal{J}$ when restricted to $\mathcal{E}$. Roughly speaking, the existence of a non-zero (on $\mathcal{E}$) solution to (9.4) would imply that every solution of the Einstein equations admits a Killing vector field. This is of course false. "Almost every" solution of the vacuum Einstein equations is devoid of symmetry [10]. Thus we expect that (9.4) has no non-trivial (*i.e.*, non-vanishing on $\mathcal{E}$) solutions. This can be proven directly using the adapted jet variables of §4, but we will not try to do it here (see [2] for details). We conclude, again, that $\rho_{ab} = 0$ and hence there are no conserved 2-forms associated to the vacuum field equations.

Despite the negative result, we *have* learned something. We see that the Einstein equations *would* admit a 2-form conservation law if every solution admitted a Killing vector field, more precisely, if (9.4) admits non-zero solutions. This result generalizes to any Lagrangian field theory via the following "rule of thumb". For a Lagrangian field theory in n-dimensions to admit lower-degree conservation laws (*i.e.*, p-form conservation laws with $p < n - 1$), the field theory must (i) be a gauge theory, *i.e.*, admit a set of generalized symmetries which are constructed from arbitrary functions, (ii) each solution of the field equations must admit a *gauge symmetry*, *i.e.*, a gauge transformation that leaves

the solution invariant. Thus, while a Lagrangian field theory must admit symmetries of the Lagrangian in order to allow for conserved $(n-1)$-forms, it must admit gauge symmetries of solutions in order to allow for lower-degree conservation laws. This rule of thumb can be made quite rigorous and precise. For a derivation of this result in the context of the BRST-antifield formalism, and some illustrative examples, see [1]. A derivation of these results in the spirit of the discussion above, a variety of generalizations and examples, and a constructive procedure for finding the conserved p-forms from solutions to the integrability conditions will be provided in [2].

Let us revisit our Lagrangian field theory examples in light of the above remarks. To begin, we expect that field theories which do not admit gauge transformations will not admit lower-degree conservation laws. Thus, there are no lower-degree conservation laws for the Klein-Gordon field. This kind of no-go result can be made quite general in terms of the rank of the symbol of the field equations [2]. Roughly speaking, only gauge theories have a sufficiently degenerate symbol to allow for lower-degree conservation laws. Next, the Maxwell equations for the 1-form A admit the gauge transformation $A \rightarrow A + df$ for any function f and thus satisfy (i). Moreover, $f = $ constant defines a gauge symmetry of any solution of the Maxwell equations. Thus (i) and (ii) are satisfied in Maxwell theory and it can be shown that (2.4) is the resulting lower-degree conservation law. It can be shown that this is the only lower-degree conservation law for the source-free Maxwell equations. Turning now to non-linear gauge theories, the vacuum Einstein equations satisfy the criterion (i) above—they admit the diffeomorphism gauge transformation, but, as we have seen, fail to satisfy (ii) and thus admit no lower-degree conservation laws. Analogous results hold for the Yang-Mills equations. The Einstein-Killing equations (2.5)-(2.6) by construction satisfy both (i) and (ii) and this leads to the existence of lower-degree conservation laws such as (2.7) and (2.8). Finally, the two-dimensional dilatonic gravity models satisfy (i) because they admit the diffeomorphism gauge transformation

$$g_{ab} \rightarrow \Psi^* g_{ab} \quad \text{and} \quad \varphi \rightarrow \Psi^* \varphi,$$

where $\Psi \colon M \rightarrow M$ is a diffeomorphism and Ψ^* denotes the corresponding pull-back. Somewhat remarkably, every solution admits a diffeomorphism gauge symmetry [3]. Explicitly,

the following vector field, which is locally constructed from the metric and dilaton field,

$$X^a = \epsilon^{ab} \nabla_b \varphi$$

satisfies the Killing equations when (2.10) is satisfied. The corresponding lower-degree conservation law is (2.11).

Finally, we introduce one more example of this phenomenon which is relevant for our upcoming discussion of asymptotic conservation laws. Our example comes from the field theory obtained by linearizing the Einstein equations about a given solution. Let (M, g) be an Einstein spacetime with cosmological constant λ (which can be set to zero if desired),

$$R_{ab}[g] = \lambda g_{ab}. \tag{9.5}$$

Let h_{ab} be a symmetric tensor field. The field equations for h_{ab} are the linearized Einstein equations

$$-\nabla^c \nabla_c h_{ab} - \nabla_a \nabla_b h_c^c + 2\nabla^c \nabla_{(a} h_{b)c} - \lambda h_{ab} = 0, \tag{9.6}$$

where all the geometric data arising in this equation (derivative operator, etc.) are defined by the fixed Einstein metric $g_{ab}(x)$. Let us consider the problem of determining what, if any, lower-degree conservation laws are admitted by this linear field theory. We begin by noting that this field theory admits the gauge transformation

$$h_{ab} \longrightarrow h_{ab} + \nabla_{(a} X_{b)},$$

where $X_a = X_a[g]$ is a covector-valued function on the jet space of symmetric tensor fields and the derivative ∇ is the total covariant derivative. Of course, this gauge transformation is just the linearized remnant of the usual diffeomorphism transformation (see (8.10)) that exists for the full, non-linear Einstein equations. Next, we note that if $X^b(x)$ is a Killing vector field of $g_{ab}(x)$, then it defines a gauge symmetry in the sense described above. This suggests that the linearized theory should admit lower-degree conservation laws if the background metric admits Killing vector fields. from an analysis of the integrability conditions for lower-degree conservation laws in the linearized theory, it can be shown that the necessary and sufficient condition for a lower-degree conservation law to exist is indeed

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that there exists a Killing vector field of the fixed background metric g_{ab}. In particular, our analysis shows that the only non-trivial lower degree conservation law (in four spacetime dimensions) is of degree 2 and, up to terms that are trivial on-shell, can be written as

$$\omega[h] = \frac{1}{32\pi}\epsilon_{ijhk}\left[h^{li}\nabla_l X^j - \frac{1}{2}h^m_m\nabla^i X^j - X_l\nabla^i h^{lj} + X^i(\nabla_l h^{jl} - \nabla^j h^m_m)\right]dx^h \wedge dx^k. \quad (9.7)$$

It is straightforward, albeit lengthy, to verify that $\omega[h]$ is a closed 2-form provided g_{ij} is an Einstein metric, h_{ab} satisfies the linearized equations, and X^a is a Killing vector field. Up to on-shell exact 2-forms, this is the only lower-degree conservation law for the linearized theory. The differential form (9.7) appears in [21] where it is used to study black hole entropy. It can also be used in asymptotic regions to define energy-momentum and angular momentum of the gravitational field as we shall discuss in the next section.

To summarize, based upon the rule of thumb presented above we can see that, in general, a non-linear gauge theory will not admit lower-degree conservation laws because the generic solutions of the field equations nominally admit no symmetries due to the intrinsically non-linear nature of the field theory. It is only for Abelian gauge theories, such as Maxwell theory and linearized Einstein theory, that the field equations are linear and do allow for gauge symmetries of solutions and corresponding lower-degree conservation laws.

Having said this, one should wonder how the non-linear system of field equations (2.9) and (2.10) describing the two-dimensional dilaton gravity theory manage to allow for a lower-degree conservation law. The answer is that the lower-degree conservation law (2.11) is really only associated to the field equation (2.10), which, by itself, can be viewed as a linear equation for the dilaton field φ on a fixed spacetime (M, g).

10. Asymptotic conservation laws and local cohomology.

In nonlinear gauge theories such as Yang-Mills theory and general relativity, conserved quantities such as charge and energy-momentum are computed from the limiting values of 2-dimensional surface integrals in asymptotic regions. A famous example of this sort of conserved quantity is the ADM energy (2.12). Such *asymptotic conservation laws* are most often derived by one of two, rather distinct, methods. The oldest method relies upon the

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construction of identically conserved currents furnished by Noether's (second) theorem and subsequent extraction of an appropriate "super-potential" to define a conserved surface integral [21,22]. Unfortunately, to my knowledge, there is no intrinsic field-theoretic criterion to select the appropriate current: for *any* field theory there are infinitely many identically conserved currents that can be expressed as the divergence of a skew-symmetric super-potential, modulo the field equations. This method of finding asymptotic conservation laws is thus somewhat *ad hoc.* An alternative, more recent approach to finding asymptotic conservation laws in gauge theories is based upon the Hamiltonian formalism. In this approach, asymptotic conservation laws arise as surface term contributions to symmetry generators [23,21]. Given appropriate asymptotic conditions, the generators of asymptotically non-trivial gauge transformations are constructed as a sum of a volume integral and a surface integral in the asymptotic region. When the field equations are satisfied the volume integrals vanish. Using the usual Hamiltonian relation between symmetry generators and conserved quantities, it follows that the surface integrals are asymptotic conservation laws. The Hamiltonian approach to finding asymptotic conservation laws lacks the *ad hoc* flavor of the super-potential formalism but is somewhat indirect: to construct asymptotic conservation laws using this method one must have the Hamiltonian formalism well in hand and one must know *a priori* the general form of the putative symmetry generator in order to find the appropriate surface integral.

Here we discuss an interpretation of asymptotic conservation laws as lower-degree conservation laws (typically $(n-2)$-forms in n-dimensions) that are associated to the field equations and asymptotic structure for the allowed solutions. We have seen that, generically, we expect no lower-degree conservation laws for for non-linear gauge theories. However, if the asymptotic structure of the theory is such that, in an appropriate sense, the asymptotic behavior of the fields is governed by the linearized theory, then the asymptotic conservation laws can be viewed as lower-degree conservation laws of the linearized theory. Thus we are able to establish that asymptotic conservation laws for field theories can be viewed as arising from (asymptotically) closed differential forms canonically associated to the field equations.

To begin, let us give a general definition of an asymptotic conservation law which

appears sufficiently general to accommodate all known examples. Denote by M an asymptotic region of the n-dimensional spacetime manifold and denote by φ^A the field of interest, which we assume is a cross section of some vector bundle or affine bundle (so that we can meaningfully take the difference between two fields). We assume the asymptotic region M is diffeomorphic to $\mathbf{R} \times (\mathbf{R}^{n-1} - C)$, where C is a compact set in $\mathbf{R}^{n-1}$. The factor of $\mathbf{R}$ represents a time axis, and the $\mathbf{R}^{n-1} - C$ factor represents space outside a compact gravitating system. Fix local coordinates $(t, x^1, \ldots, x^{n-1})$ in the asymptotic region M. We consider a fixed solution $\overset{0}{\varphi}{}^A$ to the field equations $\Delta_B[\varphi] = 0$ and then, given a second solution φ^A, we set $h^A = \varphi^A - \overset{0}{\varphi}{}^A$. Let $\omega[\overset{0}{\varphi}, h]$ be a spacetime $(n-2)$-form depending locally on the fields $\overset{0}{\varphi}{}^A$ and h^A and their derivatives. We call ω an *asymptotic conservation law for the field equations* $\Delta_B = 0$ *relative to the background* $\overset{0}{\varphi}$ if, whenever h^A satisfies the appropriate asymptotic decay conditions as $r = \sqrt{(x^1)^2 + (x^2)^2 + \ldots + (x^{n-1})^2} \to \infty$, the p-form $\omega[\overset{0}{\varphi}, h]$ satisfies

$$\omega \sim O(1) \tag{10.1}$$

and

$$D\omega \sim O(1/r). \tag{10.2}$$

Under these conditions, by Stokes theorem, the limit

$$Q[\overset{0}{\varphi}, h] = \lim_{r \to \infty} \int_{S_{(r,t_0)}} \omega[\overset{0}{\varphi}, h], \tag{10.3}$$

where $S_{(r,t_0)}$ is an $(n-2)$-dimensional sphere of coordinate radius r in the $t = t_0$ hypersurface, exists and is independent of t_0. More generally, granted the asymptotic conditions guarantee the vanishing of the integral of the right hand side of (10.2) over the timelike cylinder $\mathbf{R} \times S^2$ "at infinity", Q in (10.3) is conserved.

One way to construct asymptotic conservation laws is by, in effect, mapping $(n-2)$-form conservation laws of the linearized theory into the asymptotic structure of the theory. A general construction is described in [2,24]. Here, for reasons of brevity, I will just sketch the idea in the context of our running example, the vacuum Einstein equations in 4 dimensions. Let $\overset{0}{g}_{ij}$ denote a fixed fiducial Einstein metric on the asymptotic region relative to which decay conditions on solutions g_{ij} of the Einstein equations are specified.

In order to construct asymptotic conservation laws for these equations, we first note that $\omega[h]$ in (9.7) is closed by virtue of the identity

$$d\omega[h] = K_{ij}\nabla_k \alpha^{ijk} - \nabla_k K_{ij}\,\alpha^{ijk} + L_{ij}[h]\,\beta^{ij}, \qquad (10.4)$$

where $K_{ij} = \nabla_{(i}X_{j)}$, $L_{ij}[h]$ is the linearization of $G_{ij} + \Lambda g_{ij}$ at $g = \overset{0}{g}(x)$,

$$\alpha^{ijk} = \left[h^{ij}\overset{0}{g}^{kl} + h^{kl}\overset{0}{g}^{ij} - h^{jl}\overset{0}{g}^{ik} - h^{ik}\overset{0}{g}^{jl} + (\overset{0}{g}^{ik}\overset{0}{g}^{jl} - \overset{0}{g}^{ij}\overset{0}{g}^{kl})h_m^m \right]\Omega_l$$

$$\beta^{ij} = 2X^{(i}\overset{0}{g}^{j)l}\,\Omega_l, \qquad \text{and} \qquad \Omega_l = \frac{1}{3}\varepsilon_{abcl}\,dx^a \wedge dx^b \wedge dx^c.$$

Note that we are now denoting the fixed, fiducial metric in the asymptotic region as $\overset{0}{g}$, and

$$\nabla_a \overset{0}{g}_{bc} = 0.$$

The key point now is to re-interpret the identity (10.4) in terms of asymptotic structure. Define $h_{ij} = g_{ij} - \overset{0}{g}_{ij}$. By (10.4) and Stokes theorem we now have the following existence principle for asymptotic conservation laws: Granted that the asymptotic behavior of X^i, h_{ij}, K_{ij} and $L_{ij}[h]$ are such that (10.5), shown below, is finite and the integrals of the right-hand side of (10.4) vanish asymptotically, then

$$Q[\overset{0}{g}, h] = \lim_{r\to\infty}\int_{S_{(r,t)}}\omega[\overset{0}{g}, h], \qquad (10.5)$$

is conserved, that is, $Q[\overset{0}{g}, h]$ is independent of t. In particular, Q is conserved whenever X^i is a Killing vector of the background $\overset{0}{g}_{ij}$ and the decay conditions are such that the linearized Einstein equations for h_{ij} are satisfied at an appropriate rate at infinity.

For example, when $n = 4$, $\overset{0}{g}_{ij} = \eta_{ij}$ and $X = \partial/\partial t$, a straightforward calculation shows that (10.5) reduces to the ADM energy

$$Q[\eta, h] = \lim_{r\to\infty}\frac{1}{16\pi}\int_{S_{(r,t)}}\varepsilon_{bcd}(\partial_a h^{ab} - \partial^b h^{aa})dx^c \wedge dx^d, \quad a,b,c,d = 1,2,3,$$

which is conserved given asymptotically flat fall-off conditions, e.g.,

$$h_{ij} = O(1/r)$$

$$h_{ij,k} = O(1/r^2)$$

$$h_{ij,kl} = O(1/r^3).$$

This method of constructing asymptotic conservation laws has a number of desirable field-theoretic properties.

(1) First and foremost, it yields results in agreement with those obtained using Hamiltonian methods in the asymptotically flat and asymptotically anti-De Sitter contexts. In particular, the full complement of conservation laws available for these asymptotic structures can be obtained by letting X range over the Killing vectors of $\overset{0}{g}$ in each case.

(2) The construction is manifestly coordinate independent in the sense that the conservation laws are obtained by integrating spacetime differential forms, such as (9.7), which are defined in a coordinate independent manner.

(3) The charge Q is unchanged by the substitution $h_{ab} \to h_{ab} + \nabla_{(a}V_{b)}$ for any spacetime co-vector field V_a. This is crucial for establishing the "gauge invariance" of the conservation law.

(4) The construction readily generalizes to other field theories, for example, it is straightforward to extend our results to the Einstein-Yang-Mills equations, string-generated gravity models (Lovelock gravity), etc.

(5) The construction is not tailored to any specific asymptotic structures. In principle one can use the fundamental identity (10.4) to find asymptotic conservation laws for a wide variety of background metrics $\overset{0}{g}$ and fall-off conditions and, conversely, to analyze which fall-off conditions admit conservation laws. This feature also arises with conserved currents: Noether's theorem gives a formula for a divergence-free vector field, independently of any boundary conditions. Whether or not the conserved current actually defines a conserved quantity depends upon boundary conditions.

(6) The $(n-2)$-form $\omega[h]$ is constructed directly from the field equations with no reference made to the Bianchi identities of the theory or to any Lagrangian or Hamiltonian. A general formula for $\omega[h]$ appropriate to second-order field equations can be found in [24].

Thus asymptotic conservation laws can be viewed as a manifestation of local cohomology associated to the field equations and asymptotic conditions that are imposed.

11. Slogans.

I would like to conclude this introductory survey with a few slogans that, to some extent, summarize the main ideas we have discussed. As with all slogans, they should be taken with a grain of salt. Slogans are no substitute for rigorous theorems, but they do help with the intuition.

The theme we have encountered is an old one: symmetries and conservation laws. We have seen that conserved currents depend upon symmetries of the field equations for their existence. In particular, the lack of adequate symmetries of the vacuum Einstein equations prevents the existence of conservation laws arising as volume integrals of local densities. Conserved quantities can also be obtained by integrating locally constructed quantities over lower-dimensional submanifolds. The existence of such conservation laws depends upon the existence of gauge transformations that preserve solutions, a property we called a "gauge symmetry". Thus, while conserved $(n-1)$-forms are tied to symmetries of field equations, conserved forms of lower degree are tied to gauge symmetries of solutions to the field equations. In particular, because almost all solutions of the vacuum Einstein equations do not admit a symmetry, it follows that there are no lower-degree conservation laws for the vacuum equations. These results for the vacuum field equations of general relativity are quite typical of non-linear gauge theories. Finally, given appropriate asymptotic structure, it is possible for the solutions to the field equations to admit an asymptotic symmetry. This leads to the existence of asymptotic conservation laws via asymptotically closed $(n-2)$-forms.

Thus a cornerstone of physics, the theory of symmetry and conservation laws, is neatly described via local cohomology in field theory.

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